

An inverse method of designing cylindrical cloaks without knowing coordinate transformation

C.-W. Qiu^{1,2}, A. Novitsky³, and M. Sojačić¹

¹*Research Laboratory of Electronics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA.*

²*Department of Electrical and Computer Engineering, National University of Singapore, Kent Ridge, Singapore 119620, Republic of Singapore*

³*Department of Theoretical Physics, Belarusian State University, Nezavisimosti Avenue 4, 220050 Minsk, Belarus.*

Corresponding authors. E-mail: cwq@mit.edu

An inverse way to define the parameters of ideal cylindrical cloaks is developed, in which the interconnection between the parameters is revealed for the first time without knowing a specific coordinate transformation. The required parameters are derived in terms of the integral form of *cloaking generators*, which is very general and allows us to examine the significance of the parametric profiles. The validity of such inverse way and the invisibility characteristics are presented in full-wave numerical simulation of plane wave scattering by cloaked cylinders. © 2009 Optical Society of America

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Designing invisibility cloaks has received increasing interest from both engineering and scientific communities. Coordinate transformation is commonly used to control electromagnetic fields and render an object invisible to electromagnetic radiation [1,2]. This approach is generalized from the cloaking in terms of the conductivity [3], and further applied to acoustics and electromagnetics [4–6], which provides the possibility of concealing not only passive but also active objects within the interior [7,8].

Given a virtual space with unit parameters, coordinate transformation performs the transition from the non-scattering virtual space to the realistic physical space, which in turn does not interact with the incidence. It is due to the fact that coordinate transformation method relies on the invariance of Maxwell's equations throughout the spatial transformation. This method can thus be applied to other classical waves (e.g., pressure wave [9] and surface liquid wave [10]) excluding elastic wave [11] provided that the symmetry invariance of the

corresponding wave equation is maintained under the coordinate transformation. Therefore, traditional cloaks need to be anisotropic and inhomogeneous. The anisotropy and the inhomogeneity of the cloak, as the deformation of the space, bend the wavefront around the cloaked object and enable waves to emerge on the other side along the propagation direction without any disturbance. Based on this transformation technique, cylindrical cloaks with circular [12], elliptical [13] and arbitrary cross-sections [14] have been studied, and practical attempts to realize the cylindrical cloak have been made with promising experimental results in microwave [15] and optical [16] regimes. Three-dimensional spherical electromagnetic cloak has been studied analytically by Mie theory [7] for first-order transformation and Von Neumann's method [17] for higher-order transformation, respectively. The realization of electromagnetic spherical cloaks by multilayered isotropic coatings has been proposed [19].

Traditionally, in order to derive the anisotropic parameters for the previous cloaks, one has to know the coordinate transformation beforehand. On the contrary, in this letter, we present a novel methodology to determine the required cloaking parameters for cylindrical cloaks without knowing specific coordinate transformations. Once any parameter is given, the rest parameters can be derived in integral form associated with the introduced cloaking generator. In this method, it is not even necessary to know the exact profile of the first given parameter. The cloaking generator together with the boundary condition, in fact, replace the corresponding spatial transformation. Full-wave simulation results are provided for verification. This work is an important step forward in the search of desirable cylindrical cloaks via parametric profiles.

For impedance matching purpose, both the relative permittivity and permeability tensors of the cylindrical cloak are assumed to be equal, i.e., $\bar{\epsilon}(r) = \bar{\mu}(r) = \bar{\zeta}(r) = \zeta_r(r)\hat{r}\hat{r} + \zeta_\varphi(r)\hat{\varphi}\hat{\varphi} + \zeta_z(r)\hat{z}\hat{z}$.

First of all, we only know that a cylindrical cloak is designed by compressing the virtual region $r' < b$ into the physical region $a < r < b$ via an unknown prescribed transformation, where the prime corresponds to the virtual space. The variables of a and b are the inner and outer radius of the cylindrical cloak in the physical space, respectively. It is assumed that the spatial compression is only with respect to the radial direction, and thus the Jacobian matrix is diagonal though we still have no information of the specific form of the coordinate transformation. As a result, one can obtain

$$\bar{\zeta}(r) = \begin{pmatrix} \zeta_r(r) & 0 & 0 \\ 0 & \zeta_\varphi(r) & 0 \\ 0 & 0 & \zeta_z(r) \end{pmatrix} = \begin{pmatrix} \lambda_r/(\lambda_\varphi\lambda_z) & 0 & 0 \\ 0 & \lambda_\varphi/(\lambda_r\lambda_z) & 0 \\ 0 & 0 & \lambda_z/(\lambda_r\lambda_\varphi) \end{pmatrix}, \quad (1)$$

where

$$\lambda_r = \frac{dr}{dr'}, \quad \lambda_\varphi = \frac{r}{r'}, \quad \lambda_z = 1 \quad (2)$$

denote three principal stretches of the Jacobian matrix. Then three equations can be derived:

$$\zeta_r(r)\zeta_\varphi(r) = 1, \quad \zeta_r(r)\zeta_z(r) = \frac{r'^2}{r^2}, \quad \frac{dr'}{dr} = \sqrt{\zeta_\varphi(r)\zeta_z(r)}. \quad (3)$$

By manipulating Eq. (3) and eliminating the term $\zeta_\varphi(r)$, we derive the differential equation regarding radial and transverse parameters

$$r\sqrt{\zeta_r(r)\zeta_z(r)} \frac{d \left[r\sqrt{\zeta_r(r)\zeta_z(r)} \right]}{dr} = r\zeta_z(r). \quad (4)$$

Integrating Eq. (4), one has

$$r^2\zeta_z(r)\zeta_r(r) = C + \int_a^r 2r_1\zeta_z(r_1)dr_1, \quad (5)$$

where C is the integration constant. Due to the spatial compression ($r' = 0$ when $r = a$), it can be seen in Eq. (3) that $\zeta_r(r)\zeta_z(r) = 0$, resulting in $C = 0$. Another requirement ($r' = b$ when $r = b$) leads to the normalization condition for the z -component parameter

$$b^2 = \int_a^b 2r_1\zeta_z(r_1)dr_1, \quad (6)$$

which plays an important role in finding cloaking parameters. Here, we introduce *cloaking generator* $g(r)$ proportional to $\zeta_z(r)$, i.e., $g(r) = C_0\zeta_z(r)$ where C_0 is an arbitrary constant. Then ζ_z can be presented in the form

$$\zeta_z(r) = \frac{b^2 g(r)}{2 \int_a^b r_1 g(r_1) dr_1} \quad (7)$$

Radial and azimuthal parameters can be expressed as

$$\zeta_r(r) = \frac{2 \int_a^r r_1 g(r_1) dr_1}{r^2 g(r)}; \quad \zeta_\varphi(r) = \frac{r^2 g(r)}{2 \int_a^r r_1 g(r_1) dr_1}. \quad (8)$$

Only after all parameters are determined can corresponding coordinate transformation for such cylindrical cloaks be found, which is in contrast to the existing design approaches of transformation based cloaks:

$$r' = b \sqrt{\frac{\int_a^r r_1 g(r_1) dr_1}{\int_a^b r_1 g(r_1) dr_1}}. \quad (9)$$

One may consider some interesting situations. When two permittivities out of three are equal (i.e., $\zeta_r = \zeta_\varphi$, or $\zeta_z = \zeta_\varphi$, or $\zeta_r = \zeta_z$), it can be shown that in each situation only the trivial cloak ($r' = r$) is possible. Thus, ideal cylindrical cloak has to be realized for three different permittivities: $\varepsilon_r \neq \varepsilon_\varphi \neq \varepsilon_z$ [15, 18]. This is different from the case of spherical cloaks with only two different parameters [1, 19].

Table 1. Cloaking parameters under different power cloak generators for $\zeta_z(r)$ profile. The symbol n is an arbitrarily positive constant. We introduce $T_n(r) = (r - a)(a + r + nr)/(2 + 3n + n^2)$ to simplify the expressions corresponding to the second power profile.

generator profile	$\zeta_z(r)$	$\zeta_r(r)$	$\zeta_\varphi(r)$	implied transformation
power $g_1(r) = r^n$	$\frac{b^2 r^n (n+2)}{2(b^{n+2} - a^{n+2})}$	$\frac{2(r^{n+2} - a^{n+2})}{r^{n+2}(n+2)}$	$\frac{r^{n+2}(n+2)}{2(r^{n+2} - a^{n+2})}$	$b\sqrt{\frac{r^{n+2} - a^{n+2}}{b^{n+2} - a^{n+2}}}$
power $g_2(r) = (r - a)^n$	$\frac{b^2 (r-a)^n}{2(b-a)^n T_n(b)}$	$\frac{2T_n(r)}{r^2}$	$\frac{r^2}{2T_n(r)}$	$b \left(\frac{r-a}{b-a}\right)^{n/2} \sqrt{\frac{T_n(r)}{T_n(b)}}$
power $g_3(r) = (r - a)^n / r$	$\frac{b^2 (n+1)(r-a)^n}{2r(b-a)^{n+1}}$	$\frac{2(r-a)}{(n+1)r}$	$\frac{(n+1)r}{2(r-a)}$	$b \left(\frac{r-a}{b-a}\right)^{(n+1)/2}$

In $g_3(r)$, it is simply Pendry's classic cylindrical cloak when $n = 1$.

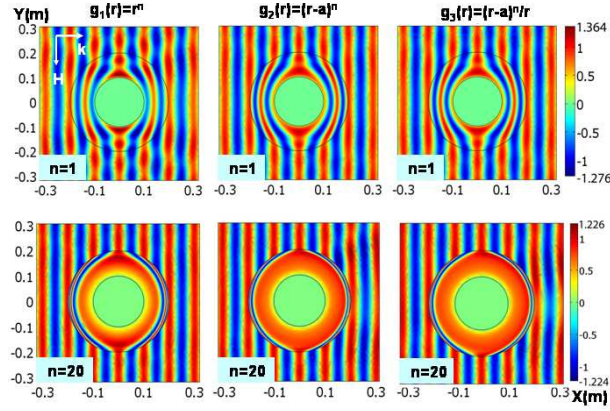


Fig. 1. (Color online) The total electric field $\text{Re}[E_z]$ on the x-y plane when $n = 1$ and $n = 20$ for the power cloak generators. The plane wave is propagating along x-axis and its electric field is polarized along z-axis. The working frequency is 3GHz. The inner rod is a PEC at the radius of $a = \lambda$, and the outer radius is $b = 2\lambda$.

To demonstrate the implementation of the cloaking generator and the proposed inverse design method, we just consider and focus on three different profiles of the power generators as given in Table 1. Certainly, one may design other profiles for the cloaking generator (e.g., parabolic, periodic, exponential, and even sinh profiles), which is out of the scope of the current letter.

When $n = 0$, g_1 and g_2 power cloak generators are identical, and their ζ_z components are constant. As an numerical example, we present two sets of near-field patterns for the power generators in Table 1, i.e., $n = 1$ and $n = 20$. In Fig. 1, it is obvious that at $n = 1$, g_1 power cloak exhibits larger disturbance in the near-field pattern not only in the wavefront but also in the variation of magnitude, and both g_2 and g_3 outperform g_1 under the same condition

(the meshing of $n = 1$ is the same for g_1 , g_2 , and g_3) simulated in COMSOL Multiphysics. If the meshing can be extremely fine, there will be no disturbance for all, while it would require too much computer memory in simulation. This non-ideality of the simulation conditions can be considered as a non-ideal cloak consisting of a number of discrete cylindrical layers. Also, at extremely high order (e.g., $n = 20$), g_1 outperforms g_2 and g_3 since it results in smaller forward scattering. The field distribution oscillates severely near the outer radius in g_2 and g_3 power cloaks ($n = 20$), implying that the meshing has to be finer in this region. It is noted that the field inside the cloak is more homogeneous at $n = 20$ because at larger ns the rays inside the cloaking shell will propagate very close to the outer radius. From the transform point of view, if the order of the power cloak generator becomes larger, more portions of the original virtual space is transformed into the region close to the outer boundary of the physical space. In theory, the cloaking performance remains while the field distribution inside the cloak is more squeezed into the outer radius. Thus, it needs much finer meshing close to the outer radius, and the cases of $n = 20$ all adopt the same fine mesh. However, we cannot make the mesh extremely fine due to the limitation of computer memory. Moreover, considering that the order n can be any arbitrary value, there may exist an optimal range of n in each power cloak in which the invisibility performance will be further improved when the optimization is applied to the discretized model [20], i.e., the calculated total scattering width becomes smaller than that of Pendry's classic one under the same computational criteria. In other cloak designs other than power cloaks, it is possible to find such optimal cases as well. Although these characteristics are of technical importance, in our letter, we just focus on the proposed inverse method which is useful to design cloak generators and determine the optimal cloak in practice.

In this letter, we reported an inverse method to derive the cloaking parameters for cylindrical cloaks which does not need to know the required coordinate transformation first. It provides us larger degree of freedom in the design of cylindrical cloaks in practice. Instead of considering the coordinate transformation, we can directly envisage the importance of the profile of just one parameter, based on which the rest can be analytically and uniquely determined. The numerical result confirms the validity of the proposed approach, and also brings the community's attention to an alternative design method for various cylindrical cloaks.

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